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On the profinite topology of right-angled Artin groups

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Abstract

In the present work, we give necessary and sufficient conditions on the graph of a right-angled Artin group that determine whether the group is subgroup separable or not. Moreover, we investigate the profinite topology of $F_2 \times F_2$ and we show that the profinite topology of the above group is strongly connected with the profinite topology of F_2 .

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1. Introduction

Subgroup separability is an extremely powerful property of groups with many topological implications. As shown by Thurston, subgroup separability allows certain immersions to lift to an embedding in a finite cover. Scott in [19] showed that subgroup separability is inherited by subgroups and finite extensions. Although free products of subgroup separable groups are subgroup separable, the same is not true for direct products. This is one of the motivations for the present work.

On the other hand, although right-angled Artin groups are known for some time (see [7,9]), they recently attracted special attention. Bestvina and Brady [3] used the kernels of their epimorphisms to \mathbb{Z} to construct examples of groups with strange finiteness properties amongst other things.

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Charney and Davis [6] and Meier and VanWyk [14] constructed, from the graph G , a cubical complex (CW-complex) and they proved that it is in fact the Eilenberg–MacLane space of G . Hsu and Wise [10] showed that G is a Coxeter group and Papadima and Suciu [18] calculated various algebraic invariants for G including the lower central series quotients. Also, Meier, Meinert and VanWyk [13] determined their geometric invariants introduced by Bieri, Neumann and Strebel.

In the present paper we study the profinite topology of a right-angled Artin group G and we show that one can decide if G is subgroup separable or not by just examining its graph. Moreover, we show that the only obstructions for G to be subgroup separable are the two well known examples of non-subgroup separable groups $F_2 \times F_2$ and L (see [12] and [17] respectively). This was the motivation to study the profinite topology of $F_2 \times F_2$ and of the *BKS* group (see [5]) which is responsible for the non-subgroup separability of L . It turned out that, for the $F_2 \times F_2$ case, the problem of determining all closed subgroups in its profinite topology is equivalent to determining the residual finiteness of every finitely presented group. Nonetheless, the positive result is that all finitely presented subgroups of $F_2 \times F_2$ are closed in the profinite topology of $F_2 \times F_2$. In fact our results show that the profinite topology of $F_2 \times F_2$ is strongly connected with that of F_2 .

2. Notation and definitions

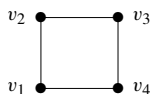
In this section we establish notation and we review some basic definitions and results.

By a graph X we mean a finite simplicial graph with vertex set VX and edge set EX . The full subgraph Y of X is a graph whose vertex set is a subset of VX , two vertices in Y being adjacent in Y if and only if they are adjacent in X . So the full subgraphs of a graph X are uniquely determined by their vertex sets. In the sequel, by a subgraph Y of a graph X , we mean the full subgraph of X , defined by VY .

If X is a connected graph, we make VX a metric space by assuming that the length of each edge is 1. So, a full subgraph Y of X is a path of length n , if Y is the graph



If $v_1 = v_{n+1}$ we say that Y is a *closed path* of length n . By a *square* we mean a closed path of length 4.



A *homeomorphism* between graphs is a simplicial function that is one-to-one on both vertices and edges and preserves adjacency.

Let X be a finite simplicial graph. The *graph group* or the *right-angled Artin group* $G(X)$ (or G for simplicity) is given by the presentation with a generator g_i for every vertex v_i of X and a defining relation $[g_i, g_j] = 1$ for each edge between vertices v_i and v_j in X .

Let X be a graph and $G(X)$ its right-angled Artin group. Let also Y be a subgraph of X . Then we can also define the right-angled Artin group of Y , $G(Y)$ and it is obvious that there is a natural embedding $G(Y) \rightarrow G(X)$. Hence, without loss of generality, from now on we will consider $G(Y)$ as a subgroup of $G(X)$.

The *profinite topology* of G is the topology whose base of closed sets consists of the finite index normal subgroups of G . Given the profinite topology, G is of course a topological group (the group operations are continuous) and it is residually finite if and only if it is Hausdorff (the trivial subgroup is closed with respect to the profinite topology). A subset $H \subseteq G$ is *separable* in G if it is closed in the profinite topology of G . One can easily show that if $K < H < G$ with $|H : K| < \infty$ then if K is closed in the profinite topology of G , so is H .

A group G is called *cyclic subgroup separable* (or π_c) if every cyclic subgroup of G is closed in the profinite topology of G . A group G is called *subgroup separable* (or *LERF*) if all its finitely generated subgroups are separable. Moreover, every subgroup of a subgroup separable group is subgroup separable [19]. Subgroup separability is a “rare” property of groups. A list of known subgroup separable groups can be found in [8].

On the other hand, non-subgroup separability is also difficult to prove. We give here two well-known examples of non-subgroup separable groups that play a major rôle in the sequel.

By L we denote the group with presentation

$$L = \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle.$$

L was shown to be non-subgroup separable by Niblo and Wise in [17]. In fact, it was shown that L contains a subgroup isomorphic to an index two subgroup of the famous example of Burns, Karrass and Solitar [5], the group with presentation

$$BKS = \langle t, a, b \mid [a, b] = 1, tat^{-1} = b \rangle.$$

The second example is older. If F_2 denotes the free group of rank two then the group $F_2 \times F_2$ was shown by Michailova (see [12]) to have non-solvable generalized word problem. Consequently, $F_2 \times F_2$ is not subgroup separable.

Finally, let $f : G \rightarrow G$ be an automorphism of G . Then $\text{Fix}(f) = \{g \in G \mid f(g) = g\}$. Obviously, $\text{Fix}(G)$ is a subgroup of G .

3. Subgroup separability

All right-angled Artin groups are residually finite by the work of Green [9] and linear by the work of Humphries [11]. In fact they are \mathbb{Z} -linear by the work of Hsu and Wise [10] and Brown [4].

Theorem 1. *All polycyclic subgroups of a right-angled Artin group G are closed in the profinite topology of G . In particular, G is cyclic subgroup separable.*

Proof. Let G be a right-angled Artin group. Then G is linear and in fact, G is a subgroup of $\text{GL}(n, \mathbb{Z})$. Hence, by [20, Corollary 1, p. 26], every soluble subgroup of $\text{GL}(n, \mathbb{Z})$ is polycyclic and so is every soluble subgroup of G . But all polycyclic subgroups of $\text{GL}(n, \mathbb{Z})$ are closed in the profinite topology of $\text{GL}(n, \mathbb{Z})$ (see [20, Theorem 5, p. 61]). Therefore, every polycyclic subgroup of G is closed in the subspace topology of G which is coarser than the profinite topology of G . Consequently, every cyclic subgroup of G is closed in the profinite topology of G , so G is cyclic subgroup separable. \square

Lemma 1. *Let G be a right-angled Artin group with graph X . If X has a path of length three as a subgraph then G is not subgroup separable.*

Proof. If X has a subgraph Y homeomorphic to a path of length three then $G(Y)$ is isomorphic to L and so G has a subgroup isomorphic to L and hence cannot be subgroup separable. \square

Lemma 2. *Let G be a right-angled Artin group with graph X . If X has a subgraph T which is a closed path of length four or more, then G is not subgroup separable.*

Proof. If T has length five or more then T contains a subtree with a path of length at least three and so T and hence X contain a subgraph homeomorphic to a path of length three thus G cannot be subgroup separable by Lemma 1.

Else, the subgraph T is homeomorphic to a square with vertices v_a, v_b, v_c, v_d . Then the right-angled group $G(T)$ is the subgroup of G generated by $\langle a, b, c, d \rangle$, with presentation

$$G(T) = \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = [d, a] = 1 \rangle = \langle a, c \rangle \times \langle b, d \rangle,$$

hence $G(T)$ is isomorphic to $F_2 \times F_2$ where F_2 is the free group of rank two. This last group is well known to be non-subgroup separable by the work of Michailova [12]. \square

Lemma 3. *Let G be a right-angled Artin group with connected graph X . If v_a is a vertex of X connected to every other vertex of X then $G = R \times \langle a \rangle$, where R is the right-angled Artin group with graph the full subgraph of X with vertex set $VX \setminus \{v_a\}$.*

Proof. Since v_a is connected to every other vertex of X , we have $G = R \times \mathbb{Z}$ where R is the subgroup of G generated by all the generators of G but a . Obviously, R contains the relations of G that do not involve a . So, in graph theoretic language, R involves all vertices of X but v_a as well as all edges of X but those that connect vertices to a . Hence, R is the subgroup of G that corresponds to the subgraph with vertex set $VX \setminus \{v_a\}$. \square

Theorem 2. *Let G be a right-angled Artin group with graph X . Then G is subgroup separable if and only if X does not contain a subgraph homeomorphic to either a square or a path of length three.*

Proof. Without loss of generality we may assume that X is connected. If X is disconnected we work with the connected components of X . The subgroup separability of G is then a consequence of the fact that the free product of two subgroup separable groups is subgroup separable.

Assume first that X does not contain a subgraph homeomorphic to either a square or a path of length three. We use induction on the number of vertices of X .

If X contains one or two vertices then G is isomorphic to either \mathbb{Z} or \mathbb{Z}^2 and so is subgroup separable. If X contains three vertices then there is at least one vertex, say v_a , that is connected to every other vertex of X . Then by Lemma 3, $G = A \times \mathbb{Z}$ where A is either a free abelian group of rank two or a free group of rank two. In both cases G is subgroup separable, in the first since it is abelian and in the second, by the work of Allenby and Gregorac [1].

Assume that every right-angled Artin group having a graph with k vertices that contains no subgraph homeomorphic to either a square or a path of length three is subgroup separable.

Let Y be a graph with $k + 1$ vertices that satisfies the hypotheses of the theorem. Then by lemma in [7], there is at least one vertex in Y that is connected to every other vertex of Y . So $R = M \times \mathbb{Z}$ where, by Lemma 3, M is a right-angled Artin group that corresponds to the

subgraph with vertex set $VY \setminus \{v\}$. So M is subgroup separable from the inductive hypothesis and so R is subgroup separable from Lemma 3 in [15].

Conversely, if G is subgroup separable it cannot contain a subgroup isomorphic neither to L nor to $F_2 \times F_2$. Hence, its graph X cannot have a subgraph homeomorphic to neither a square nor a path of length three. \square

We should mention here that the above theorem easily generalizes to graph groups, that is Artin groups with each vertex associated to a free abelian group of finite rank.

4. The profinite topology of $F_2 \times F_2$

The following lemma is a simple generalization of Lemma 2 in [16]. The proof is practically the same as of [16, Lemma 2] but is included here for completeness.

Lemma 4. (See [16].) *Let G be a group and let H be a finitely generated, subgroup separable, normal subgroup of G such that G/H' is subgroup separable for every characteristic subgroup H' of H . Let also M be a finitely generated subgroup of G . Then M is closed in the profinite topology of G if $M \cap H$ is closed in the profinite topology of H .*

Proof. It suffices to show that $\bigcap_{N \in \mathcal{N}} MN = M$ where \mathcal{N} is the set of all normal subgroups of finite index in G . Let \mathcal{C} be the set of all characteristic subgroups of finite index in H . For every $H' \in \mathcal{C}$ we have that G/H' is subgroup separable and that MH'/H' is finitely generated, hence

$$\bigcap_{V \in \mathcal{V}} V \frac{MH'}{H'} = \frac{MH'}{H'}$$

or equivalently

$$\bigcap_{N \in \mathcal{N}} \frac{NH'}{H'} \frac{MH'}{H'} = \frac{MH'}{H'}$$

where \mathcal{V} is the set of all normal subgroups of finite index in G/H' . Consequently, $\bigcap_{N \in \mathcal{N}} MN$ is a subset of MH' for every $H' \in \mathcal{C}$.

Now, let $U = \bigcap_{H' \in \mathcal{C}} MH'$. Obviously M is a subgroup of U . So,

$$U \cap H = \left(\bigcap_{H' \in \mathcal{C}} MH' \right) \cap H = \bigcap_{H' \in \mathcal{C}} (M \cap H)H'.$$

But

$$\bigcap_{H' \in \mathcal{C}} (M \cap H)H' = \bigcap_{N \in \mathcal{N}'} (M \cap H)N = M \cap H$$

since $M \cap H$ is closed in the profinite topology of H . In the above, \mathcal{N}' is the set of all finite index normal subgroups of H . So, $U \cap H = M \cap H$.

Let $u \in U$. Then, for every $H' \in \mathcal{C}$ there is an $h' \in H'$ and an $l' \in M$ such that $u = l'h'$. Hence, $(l')^{-1}u = h' \in H'$ and so $(l')^{-1}u \in H$. On the other hand, M is a subgroup of U and so $l' \in U$.

Therefore $(l')^{-1}u \in U$. Hence, $(l')^{-1}u \in U \cap H = M \cap H$. Thus, there is an $l_1 \in M$ such that $l^{-1}u = l_1$ which implies that $u = ll_1 \in M$. So $U \subseteq M$. But $M \subseteq U$ and therefore $U = M$. Since $\bigcap_{N \in \mathcal{N}} MN \subseteq U = M$ we have that $\bigcap_{N \in \mathcal{N}} MN = M$ as required. \square

If $C = A \times B$ then, by abusing notation, we identify $A \times \{1\}$ with A and $\{1\} \times B$ with B . So we can now prove the following.

Proposition 1. *Let $C = A \times B$ where A, B are subgroup separable groups. A finitely generated subgroup M of C is closed in the profinite topology of C if and only if $M \cap A$ (or $M \cap B$) is closed in the profinite topology of A (or B).*

Proof. If $M \cap A$ is closed in the profinite topology of A then M is closed in the profinite topology of C , by Lemma 4.

Assume now that M is closed in C . Both A and B are also closed in the profinite topology of C . Indeed, if $g \in G$ with $g \notin A$ then under the projection homomorphism $f : A \times B \rightarrow B$, $f(A) = 1$ but $f(g) \neq 1$. The result follows easily from the fact that B is subgroup separable. Hence $M \cap A$ is closed in the profinite topology of C as an intersection of closed sets. Consequently, $M \cap A$ is closed in the subspace topology of A which is coarser than the profinite topology of A . Hence, $M \cap A$ is closed in the profinite topology of A . The case $M \cap B$ is equivalent. \square

Now we can use the above proposition to show a positive and a negative result.

Let F'_2 be an isomorphic copy of F_2 , the free group of rank two. The positive result is the following.

Corollary 1. *Let H be a finitely presented subgroup of $G = F_2 \times F'_2$. Then H is closed in the profinite topology of G .*

Proof. By the work of Baumslag and Roseblade [2], H is either free or else has a subgroup of finite index that is the product $H_1 \times H_2$ with $H_1 = F_2 \cap H$ and $H_2 = F'_2 \cap H$. In the second case, each H_i , $i = 1, 2$ is finitely generated and so is closed in the profinite topology of F_2 (and F'_2) so $H_1 \times H_2$ is closed in the profinite topology of G , by Proposition 1. Consequently, H is closed in the profinite topology of G .

In the first case, let H be a free subgroup of $F_2 \times F'_2$. If either $H \cap F_2$ or $H \cap F'_2$ are trivial then H is closed in the profinite topology of $F_2 \times F'_2$ by Proposition 1. If, on the other hand, $H \cap F_2 \neq 1 \neq H \cap F'_2$ then H contains a subgroup isomorphic to \mathbb{Z}^2 , a contradiction to the hypothesis that H is free. \square

Now the negative result. The following construction is based on an idea of Michailova [12]. Let

$$H = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

be any finitely presented group and let F_n be the free group on abstract generators $\langle x_1, \dots, x_n \rangle$. Obviously, $F_n \times F_n$ can be considered as a finite index subgroup of $F_2 \times F'_2$, so every subgroup of $F_n \times F_n$ is closed in the profinite topology of $F_n \times F_n$ if and only if it is closed in the profinite topology of $F_2 \times F'_2$.

Let L_H be the subgroup of $F_n \times F_n$ generated by

$$L_H = \langle (x_i, x_i), i = 1, \dots, n, (1, r_j), j = 1, \dots, m \rangle.$$

Then $L_H \cap F_n$ is the normal closure of $\langle r_j, j = 1, \dots, m \rangle$ as a subgroup of F_n . So, by Proposition 1, L_H is closed in the profinite topology of $F_n \times F_n$, if and only if $L_H \cap F_n$ is closed in the profinite topology of F_n or equivalently if and only if the group

$$H = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

is residually finite. So we have the following.

Corollary 2. *The problem of determining all closed finitely generated subgroups of $G = F_2 \times F_2'$ with respect to the profinite topology is equivalent to the problem of determining the residual finiteness of all finitely presented groups.*

This last corollary is in accordance with the work of Stallings [21] which shows that all kinds of “nasty” subgroups can occur in $F_2 \times F_2'$.

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